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On Slow Decay of the Solution of the Initial-Boundary Value Problem for the Wave Equation in the Exterior of a Three-Dimensional Obstacle

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Submitted by Peter Lax

INTRODUCTION

The decay of solution of the initial-boundary value problem for the wave equation in the exterior of a three-dimensional obstacle has been the subject of many papers. Here we shall not give a full account of all the researches in this area but only quote few which are directly related to this paper.

Lax, Morawetz, and Phillips [1] (later refered as LMP) showed that the solution of the wave equation in the exterior of a three-dimensional bounded smooth star-shaped body decays exponentially when the boundary condition is of Dirichlet type and the smooth initial data have compact support. By simply estimating the well-known solution of Oberhettinger [2], Zachmanoglou [3] has shown that the result of LMP is not true when the obstacle is an infinite wedge of arbitrary angle. Since the wedge as an obstacle has at least two of its three dimensions being extended to infinity and also has a corner, it is not known whether the slow decay is caused by the infinite boundary or by the corner.

It is the purpose of this paper to show that the result of LMP is not generally true even when one of the dimensions of a three-dimensional smooth bounded star-shaped obstacle is extended to infinity. This is achieved by first choosing an infinitely long circular cylinder as the obstacle, then constructing the solution upon using the method of separation of variables, and finally investigating the solution for large time variable.

SLOW DECAY OF THE SOLUTION

Let D be the exterior region of an infinitely long circular cylinder of radius a in the three-dimensional space and B be its boundary. Let $\mu(\bar{r}, t)$

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be the solution of the initial-boundary value problem for the wave equation in D:

$$\nabla^2 \mu - \frac{\partial^2 \mu}{\partial t^2} = 0, \tag{1}$$

$$\mu(\bar{r}, 0) = f(\bar{r}), \qquad \left.\frac{\partial \mu(\bar{r}, t)}{\partial t}\right|_{t=0} = g(\bar{r}) \tag{2}$$

and either (a)

$$\mu(\bar{r}, t) = 0 \qquad \text{for} \qquad \bar{r} \in B, \quad t \geqslant 0 \tag{3}$$

or (b)

$$\frac{\partial \mu(\bar{r}, t)}{\partial n} = 0 \qquad \text{for} \qquad \bar{r} \in B, \quad t \geqslant 0, \tag{4}$$

wher f and g are smooth functions in D and have compact supports $\mathscr{D}_f$ and $\mathscr{D}_g$, respectively.

THEOREM. *At each point in D, the solution $\mu(\bar{r}, t)$ of the initial-boundary value problems (1), (2) and (3) and (1), (2) and (4) decays like $1/t^2 (\log t)^3$ and $1/t^4$, respectively for t large enough.*

PROOF. We shall prove the first part of the theorem first. By introducing the Laplace transformation pair

$$\cup (\bar{r}, s) = \int_0^\infty e^{-st} \mu(\bar{r}, t)\, dt \tag{5}$$

and

$$\mu(\bar{r}, t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} e^{st} \cup (\bar{r}, s)\, ds, \qquad \epsilon > 0, \tag{6}$$

and letting $s = -ik$, we can transform the original system of equations into a time-independent system,

$$\nabla^2 \cup (\bar{r}, k) + k^2 \cup (\bar{r}, k) = g(\bar{r}) - ikf(\bar{r}), \tag{7}$$

$$\cup (\bar{r}, k) = 0 \qquad \text{for} \qquad \bar{r} \in B, \tag{8}$$

and radiation condition at $|\bar{r}| \to \infty$.
It is well known that the solution of (3) and (4) is

$$\cup (\bar{r}, k) = \int_{\mathscr{D}} G(\bar{r}, \bar{r}', k)\, [g(\bar{r}') - ikf(\bar{r}')]\, d\bar{r}', \tag{9}$$

where $\mathscr{D} = \max(\mathscr{D}_f, \mathscr{D}_g)$ and the Green's function $G(\bar{r}, \bar{r}', k)$ satisfies

$$\nabla^2 G(\bar{r}, \bar{r}', k) + k^2 G(\bar{r}, \bar{r}', k) = \delta(\bar{r} - \bar{r}'), \tag{10}$$

$$G(\bar{r}, \bar{r}', k) = 0 \qquad \text{for} \qquad \bar{r} \in B, \tag{11}$$

and radiation condition at $|\bar{r}| \to \infty$.

Upon using the method of separation of variables, the solution of (10) and (11) is obtained as

$$G(\bar{r}, \bar{r}', k) = \frac{-e^{ik|r-r'|}}{4\pi\,|\,\bar{r}-\bar{r}'\,|} + \frac{1}{8\pi i}\sum_{m=0}^{\infty} \epsilon_m \cos m(\phi - \phi')$$

$$\cdot \int_{l} e^{i\omega(z-z')} \frac{J_m([\sqrt{k^2-\omega^2}]a)}{H_m^{(1)}([\sqrt{k^2-\omega^2}]a)} H_m^{(1)}([\sqrt{k^2-\omega^2}]\rho)$$

$$\cdot\ H_m^{(1)}([\sqrt{k^2-\omega^2}]\rho')\, d\omega,$$

$$\epsilon_0 = 1, \qquad \epsilon_1 = \epsilon_2 = \epsilon_3 = \cdots = 2, \tag{12}$$

where $\bar{r} = (\rho, \phi, z)$ and $\bar{r}' = (\rho', \phi', z')$ (Fig. 1), and the integration path is shown in Fig. 2.

From the properties of f and g and from (6), (9), and (12), the solution of (1), (2), and (3) is

$$\mu(\bar{r}, t) = \mu_{in}(\bar{r}, t) + \sum_{m=0}^{\infty} \mu_m(\bar{r}, t), \tag{13}$$

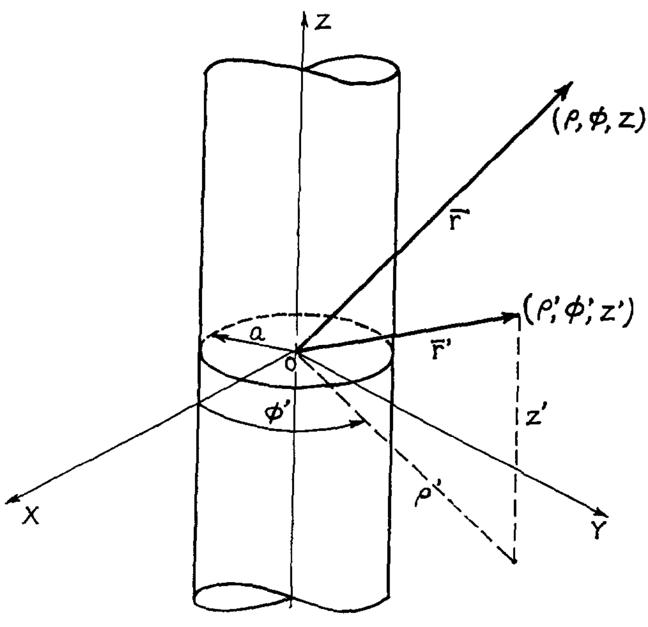

FIG. 1

where

$$\mu_{in}(\bar{r}, t) = \frac{-1}{4\pi} \int_{\mathscr{D}} \left[\frac{g(\bar{r}')}{|\bar{r} - \bar{r}'|} \delta(|\bar{r} - \bar{r}'| - t) + i \frac{f(\bar{r}')}{|\bar{r} - \bar{r}'|} \frac{\partial}{\partial t} \delta(|\bar{r} - \bar{r}'| - t) \right] d\bar{r}' \qquad (14)$$

and

$$\mu_m(\bar{r}, t) = \frac{\epsilon_m}{16\pi^2 i} \int_{\mathscr{D}} [g(\bar{r}') \, II_{m1} - if(\bar{r}') \, II_{m2}] \, d\bar{r}' \qquad (15)$$

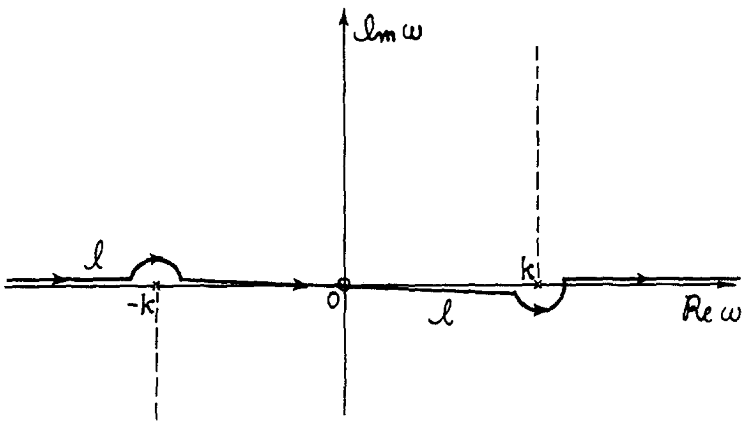

FIG. 2

with

$$II_{m1} = \int_{i\epsilon-\infty}^{i\epsilon+\infty} e^{-ikt} \int_{\ell} e^{i\omega(z-z')} \frac{J_m([\sqrt{k^2 - \omega^2}]a)}{H_m^{(1)}([\sqrt{k^2 - \omega^2}]a)} \times H_m^{(1)}([\sqrt{k^2 - \omega^2}]\rho') \, H_m^{(1)}([\sqrt{k^2 - \omega^2}]\rho) \, d\omega \, dk \qquad (16)$$

and

$$II_{m2} = \int_{i\epsilon-\infty}^{i\epsilon+\infty} ke^{-ikt} \int_{\ell} e^{i\omega(z-z')} \frac{J_m([\sqrt{k^2 - \omega^2}]a)}{H_m^{(1)}([\sqrt{k^2 - \omega^2}]a)} \times H_m^{(1)}([\sqrt{k^2 - \omega^2}]\rho') \, H_m^{(1)}([\sqrt{k^2 - \omega^2}]\rho) \, d\omega \, dk. \qquad (17)$$

From (14) it is obvious that $\mu_{in}(\bar{r}, t)$ is identically zero after a finite time. One should also notice that $\mu_{in}(\bar{r}, t)$ is the solution of (1) and (2), i.e., with the cylinder being removed.

Next we proceed to examine $\mu_0(\bar{r}, t)$ for large time. But first we have to evaluate II_{01} asymptotically for large t. To do so we must be able to interchange the order of integration freely. Unfortunately, the integrand of

II_{01} is only absolutely integrable in ω and Riemann integrable in k, therefore we cannot interchange the order of integration as it stands. However, we shall use the following technique to achieve our purpose. Since $J_0(Z)$ is an entire function of Z and $H_0^{(1)}(Z)$ has a logarithmic branch-point at $Z = 0$ and no zeros in the entire Z-plane [4] [5], the only singularities of II_{01} and II_{02} in the ω-plane are the two branch-points at $\omega = \pm k$.

If $(z - z') > 0$, we can deform the original integration path (Fig. 2) into a new path $R + L_1 + L_2$ (Fig. 3) around the branch-point at $\omega = k$.

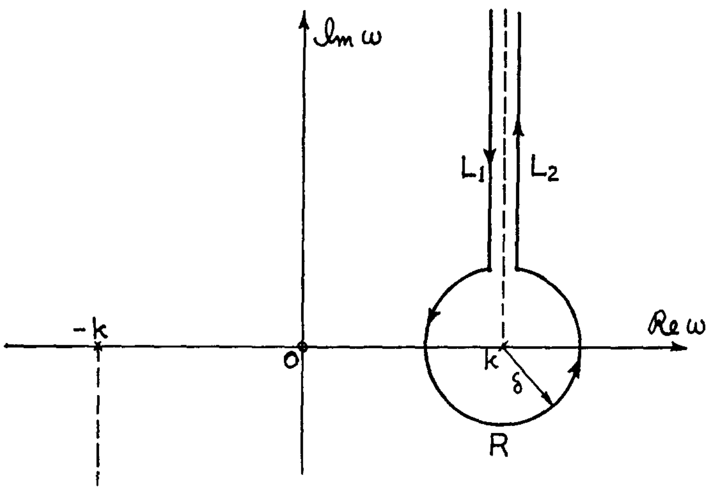

FIG. 3

Since $\omega = k + \beta e^{i(\pi/2)}$, $i = e^{i(\pi/2)}$, $-1 = e^{i\pi}$, etc., on L_1, $\omega = k + \beta e^{-i(3\pi/2)}$, $i = e^{-i(3\pi/2)}$, $-1 = e^{-i\pi}$, etc., on L_2, and the contribution come from $\lim_{\delta\to 0} \int_R$ is zero, we obtain

$$II_{01} = -i \int_{i\epsilon-\infty}^{i\epsilon+\infty} e^{-ik(t-z+z')} \int_0^\infty e^{-\beta(z-z')}$$

$$\cdot \left\{ \frac{J_0(A_1 a)}{H_0^{(1)}(A_1 a)} H_0^{(1)}(A_1\rho')\, H_0^{(1)}(A_1\rho) \right.$$

$$\left. - \frac{J_0(A_2 a)}{H_0^{(1)}(A_2 a)} H_0^{(1)}(A_2\rho')\, H_0^{(1)}(A_2\rho) \right\} d\beta\, dk, \tag{18}$$

where

$$A_1 = \sqrt{(\beta^2 e^{i2\pi} + 2k\beta e^{i(3\pi/2)})} \tag{19}$$

and

$$A_2 = \sqrt{(\beta^2 e^{-i4\pi} + 2k\beta e^{-i(5\pi/2)})}. \tag{20}$$

Because of the integrand of (18) decays like

$$e^{-\beta(z-z'+\rho+\rho'-a)} \qquad \text{and} \qquad e^{-|k|(\rho+\rho'-a)}$$

for large β and $|k|$, respectively, it is absolutely integrable for both β and k.

Hence we may interchange the order of integration. Now we observe that the new integrand of II_{01} has a branch-point at $k = -i(\beta/2)$. If $t > z - z' > 0$, we can deform the original path into a new path $R + C_1 + C_2$ (Fig. 4) around the branch-point $k = -i(\beta/2)$. To be consistent with the way of choosing branches on L_1 and L_2, we have

$$k = \frac{\beta}{2} e^{-i(\pi/2)} + \alpha e^{-i(\pi/2)}, \quad -i = e^{-i(\pi/2)}, \quad -1 = e^{-i\pi}, \quad \text{etc.} \quad \text{on} \quad C_1$$

and

$$k = \frac{\beta}{2} e^{i(3\pi/2)} + \alpha e^{i(3\pi/2)}, \quad -i = e^{i(3\pi/2)}, \quad -1 = e^{i\pi}, \quad \text{etc.} \quad \text{on} \quad C_2.$$

Since the contribution come from $\lim_{\delta\to0} \int_R$ is zero, we get

$$\begin{aligned} II_{01} = \int_0^\infty e^{-\beta(t+z-z')/2} \int_0^\infty e^{-\alpha(t-z+z')} \\ \cdot \Big\{ \frac{J_0(E_1 a)}{H_0^{(1)}(E_1 a)} H_0^{(1)}(E_1\rho') H_0^{(1)}(E_1\rho) - \frac{J_0(E_2 a)}{H_0^{(1)}(E_2 a)} H_0^{(1)}(E_2\rho') H_0^{(1)}(E_2\rho) \\ - \frac{J_0(E_3 a)}{H_0^{(1)}(E_3 a)} H_0^{(1)}(E_3\rho') H_0^{(1)}(E_3\rho) \\ + \frac{J_0(E_4 a)}{H_0^{(1)}(E_4 a)} H_0^{(1)}(E_4\rho') H_0^{(1)}(E_4\rho) \Big\} d\alpha\, d\beta \end{aligned} \tag{21}$$

where

$$E_1 = \sqrt{(2\alpha\beta e^{i\pi})}, \tag{22}$$

$$E_2 = \sqrt{(2\alpha\beta e^{-i3\pi})}, \tag{23}$$

$$E_3 = \sqrt{(2\alpha\beta e^{i3\pi})}, \tag{24}$$

and

$$E_4 = \sqrt{(2\alpha\beta e^{-i\pi})}. \tag{25}$$

Because II_{01} is in standard double Laplace transformation, from theory of Laplace transformation [6], the large time asymptotic behavior of II_{01} is determined by the asymptotic behavior of the integrand of II_{01} for small transform variables α and β. Upon using the asymptotic expansions of Bessel functions,

$$J_0(Z) = 1 + 0(Z^2) \tag{26}$$

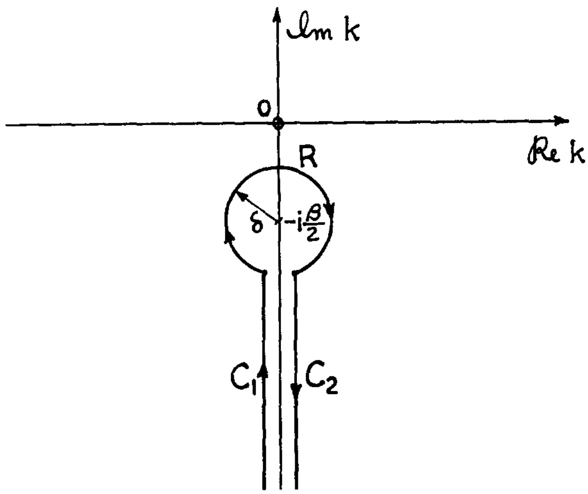

FIG. 4

and

$$H_0^{(1)}(Z) = 1 + i\frac{2}{\pi}\log\frac{\Gamma Z}{2} + 0(Z^2 \log Z), \tag{27}$$

$$\Gamma \cong 1.7811,$$

we obtain

$$\Pi_{01} = 8\pi i \log\frac{\rho}{a}\log\frac{\rho'}{a}\int_0^\infty e^{-\beta(t+z-z')/2}\int_0^\infty e^{-\alpha(t-z+z')}$$
$$\cdot\left\{\left[\log\frac{\Gamma a}{2}\sqrt{(2\alpha\beta)}\right]^{-3} + 0[(\log\sqrt{(\alpha\beta)})^{-4}]\right\} d\alpha\, d\beta. \tag{28}$$

For $t \gg z - z' > 0$, from Abelian theorem [6]

$$\Pi_{01} = -16\pi i \log\frac{\rho}{a}\log\frac{\rho'}{a}$$
$$\times \{[t^2 - (z - z')^2]^{-1}[\log(\Gamma a)^{-1}(t^2 - (z - z')^2)^{1/2}]^{-3} + 0[t^{-2}(\log t)^{-4}]\}. \tag{29}$$

Because of the extra k of Π_{02}, in this case

$$\Pi_{02} = 0[t^{-3}(\log t)^{-3}]. \tag{30}$$

Hence

$$\mu_0(\bar{r}, t) = 0[t^{-2}(\log t)^{-3}] \tag{31}$$

for $t \gg z - z' > 0$. Obvisously, with minor changes of above proof, (31) is also true for $t \gg z' - z > 0$.

By the same technique, it can be shown that all the $\mu_m(\bar{r}, t)$, $m = 1, 2, 3, \cdots$ decay no slower than $0[t^{-4}(\log t)]$. Finally we obtain

$$\mu(\bar{r}, t) = 0[t^{-2}(\log t)^{-3}]. \tag{32}$$

The proof of the second part of the theorem is exactly the same as the proof of the first part and the result is

$$\mu(\bar{r}, t) = 0(t^{-4}). \tag{33}$$

Q.E.D.

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